Journal of Mathematical Analysis and Applications **229,** 605–630 (1999)
Article ID jmaa.1998.6188, available online at http://www.idealibrary.com on IDEAL®

Singularly Perturbed Multidimensional Switching Diffusions with Fast and Slow Switchings*

G. Yin† and M. Kniazeva

Submitted by E. S. Lee

Received June 5, 1997

This work is concerned with singularly perturbed switching diffusions with fast and slow components. The diffusion component of the joint process is multidimensional and the diffusive motion evolves an order of magnitude slower than that of the jumps or the switchings. The jump component may be decomposed into several groups. There is one group consisting of transient states, and all other groups involve recurrent states. Within each group, there are rapid switchings. Among different groups, the switchings occur relatively infrequently. Under simple conditions, an asymptotic expansion of the probability density is developed; error bounds and justification of the expansion are provided. © 1999 Academic Press

1. INTRODUCTION

In this work, we concern ourselves with the asymptotic behavior of a class of singularly perturbed switching diffusion (or jump diffusion) processes. Under simple conditions, we derive an asymptotic expansion of the corresponding probability density functions.

Recently, there have been renewed interests in investigating singularly perturbed Markovian systems. One of the main motivations stems from the needs in a wide variety of applications arising in control and optimization of manufacturing systems; see [10, 11, 12] and [15, 16] among others. In [7, 8], we studied singularly perturbed Markov chains and derived asymptotic expansions by examining the corresponding forward equations. Central limit theorems, and further probabilistic structures of such systems were developed in [15, 17, 18]. The focus of these references is mainly on pure jump

*This research was supported in part by the National Science Foundation under grant DMS-9529738, and in part by Wayne State University.

†E-mail address: gyin@math.wayne.edu.

0022-247X/99 $30.00

processes (cf. [13]). The study of singularly perturbed switching diffusions was initiated in [5], in which the generator corresponding to the jump component is weakly irreducible (the definition is included in Section 2). Both fast switching and fast varying diffusion processes were considered.

This paper examines a more complex model. The jump component of the generator in our model has both fast and slow parts. The jump component may be decomposed into several groups. There is one group consisting of transient states, whereas all the other groups involve recurrent states. In addition, multidimensional diffusion processes are considered in lieu of one-dimensional diffusion processes as in the aforementioned reference. Under suitable conditions, we show that an asymptotic expansion for the probability distribution can still be constructed.

One of the motivations of the current study is also from scenarios arising in a manufacturing setting. Consider a manufacturing system involving a number of unreliable machines or units. The demand of the produced product is subject to random variations (modeled by periodic diffusions). As in many of the usual manufacturing systems, the demand changes much more slowly than that of the capacity of the machines. Furthermore, among these failure-prone machines, some of their capacities change very rapidly and the others vary slowly. Understanding the asymptotic properties of the probability distribution plays an essential role in the production planning and in obtaining any meaningful results of control and optimization of the underlying systems.

We arrange the remainder of the paper as follows. Section 2 begins with the precise formulation of the problem. Section 3 concentrates on the asymptotic expansion of the probability density. The method we are using is constructive and provides the instruction on how various terms can be obtained. Section 4 proceeds with the validation of the asymptotic expansion. Finally, we make a few more remarks in Section 5. At the end of the paper, an appendix containing the sketches of proofs of a couple of lemmas is included.

For future use, throughout the paper, the symbol $'$ denotes the transpose, K is used as a generic constant with the understanding of $K + K = K$ and $KK = K$, and $\mathbb{1} = (1, \ldots, 1)' \in \mathbb{R}^m$; similarly, $\mathbb{1}_{m_k} \in \mathbb{R}^{m_k}$.

2. PROBLEM FORMULATION

We work with a finite time horizon; assuming that there is a $T > 0$, we work with $t \in [0, T]$ throughout the paper. Let $\varepsilon > 0$ be a small parameter. Consider a nonstationary Markov process $Y^\varepsilon(t) = (X(t), \gamma^\varepsilon(t))$, where $X(t)$ is an r-dimensional diffusion process (see [14]) and $\gamma^\varepsilon(t)$ is a pure jump process (see [1]).

The state space of the process $Y^\epsilon(\cdot)$ is $\mathscr{X} = [0,1]^r \times \mathscr{M}$. We consider the case of periodic diffusions on compact space. To be more specific, the diffusion component has state space $[0,1]^r$ and the jump process has state space $\mathscr{M}$. If $\gamma^\varepsilon(t) = \alpha$, the evolution of $X(t)$ is represented by the differential operator $\mathscr{D}_\alpha$ with

$$\mathscr{D}_\alpha h(x,t) = \frac{1}{2}\sum_{i,j=1}^{r} a_{ij}^\alpha(x,t)\frac{\partial^2 h(x,t)}{\partial x_i x_j} + \sum_{i=1}^{r} b_i^\alpha(x,t)\frac{\partial h(x,t)}{\partial x_i}$$

for appropriate smooth functions $h(\cdot)$, $a^\alpha(\cdot) = (a_{ij}^\alpha(\cdot))$, and $b^\alpha(\cdot) = (b_i^\alpha(\cdot))$. The pure jump component $\gamma^\varepsilon(t)$ satisfies

$$P(\gamma^\varepsilon(t+\delta) = \beta | \gamma^\varepsilon(t) = \alpha, X(t) = x) = q_{\alpha\beta}^\varepsilon(x,t)\delta + o(\delta),$$

where $o(\delta) \to 0$ as $\delta \to 0$, for each $\alpha, \beta \in \mathscr{M}$, $q_{\alpha\beta}^\varepsilon(x,t) \ge 0$ when $\beta \neq \alpha$, and

$$q_{\alpha\alpha}^\varepsilon(x,t) = -\sum_{\beta\neq\alpha} q_{\alpha\beta}^\varepsilon(x,t).$$

Note that although the joint process is Markovian, $\gamma^\varepsilon(t)$ is generally non-Markovian. However, for fixed x, it may be considered as a Markov chain with generator $Q^\varepsilon(x,t) = (q_{\alpha\beta}^\varepsilon(x,t))$. Assume that all coefficients $b^\alpha(\cdot)$, $a^\alpha(\cdot)$, $q_{\alpha\beta}^\varepsilon(\cdot)$ are sufficiently smooth functions of (x,t). It is known (see [3]) that for smooth vector-valued function $f(x,t) = (f^1(x,t),\ldots,f^m(x,t))$, the generator L of this switching-diffusion process has the form

$$\begin{aligned}(Lf)_\alpha = {} & \frac{\partial f^\alpha}{\partial t} + \sum_{i=1}^{r} b_i^\alpha(x,t)\frac{\partial f^\alpha}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{r} a_{ij}^\alpha(x,t)\frac{\partial^2 f^\alpha}{\partial x_i x_j} \\ & + \sum_{\beta=1}^{m} q_{\alpha\beta}^\varepsilon(x,t) f^\beta(x,t),\end{aligned} \tag{1}$$

for $\alpha = 1,\ldots,m$. As a consequence, the probability density of the process, $p^\varepsilon(x,t) = (p_1^\varepsilon(x,t),\ldots,p_m^\varepsilon(x,t))$ satisfying

$$\int_\Gamma p_\alpha^\varepsilon(x,t)\,dx = P(X(t)\in\Gamma, \gamma^\varepsilon(t) = \alpha),$$

is the solution of the Kolmogorov–Fokker–Planck equation

$$\frac{\partial p_\alpha^\varepsilon}{\partial t} = \mathscr{D}_\alpha^* p_\alpha^\varepsilon + \sum_{\beta=1}^{m} p_\beta^\varepsilon q_{\alpha\beta}^\varepsilon,$$

where

$$\mathscr{D}_\alpha^* \cdot = \mathscr{D}_\alpha^*(x,t)\cdot = \frac{1}{2}\sum_{i,j=1}^{r}\frac{\partial^2}{\partial x_i x_j}(a_{ij}^\alpha(x,t)\ \cdot) - \sum_{i=1}^{r}\frac{\partial}{\partial x_i}(b_i^\alpha(x,t)\ \cdot),$$

$$p_\alpha^\varepsilon(x,0) = g_\alpha(x), \qquad \alpha = 1,\ldots,m,$$

and $g(x) = (g_1(x), \ldots, g_m(x))$ is the initial distribution for $Y(t)$. The equation above may be put in a vector form

$$\frac{\partial p^\varepsilon}{\partial t} = \mathscr{D}^* p^\varepsilon + p^\varepsilon Q^\varepsilon,$$

where

$$\begin{aligned}
\mathscr{D}^* p^\epsilon(x,t) &= (\mathscr{D}_1^* p_1^\varepsilon(x,t), \ldots, \mathscr{D}_m^* p_m^\varepsilon(x,t)), \\
p^\varepsilon(x,0) &= g(x) \text{ satisfying } \int_{[0,1]^r} g(x)\mathbb{1}\, dx = 1 \quad \text{and} \\
g_\alpha(x) &\geq 0 \quad \text{for } \alpha = 1, \ldots, m.
\end{aligned} \tag{2}$$

Remark. The state space S of the diffusion component is an r-torus (r-product of the unit circles). By identifying the end points 0 and 1, $[0,1]^r$ becomes the coordinate representation of S. The state space of the jump component is naturally divided into a number of subsets

$$\begin{aligned}
\mathscr{M} &= \left\{s_{11}, \ldots, s_{1m_1}, s_{21}, \ldots, s_{2m_2}, \ldots, s_{l1}, \ldots, s_{lm_l}, s_{*1}, \ldots, s_{*m_*}\right\} \\
&= \mathscr{M}_1 \cup \mathscr{M}_2 \cup \cdots \cup \mathscr{M}_l \cup \mathscr{M}_*,
\end{aligned}$$

and $m = m_1 + m_2 + \cdots + m_*$, i.e., the cardinality of $\mathscr{M}$ is m. The model we are interested in is that on each torus, the process behaves as a diffusion. There are m tori altogether. The random process switches rapidly from one torus to another. An even simpler but illustrative example is that $r = 1$. In this case, we have a number of circles. The process jumps frequently from one circle to the other, and behaves in a diffusive manner on each circle.

For the subsequent study, assume that

$$Q^\varepsilon(x,t) = \frac{1}{\varepsilon}\widetilde{Q}(x,t) + \widehat{Q}(x,t), \tag{3}$$

where $\widetilde{Q}(x,t)$ governs the rapidly changing part, and $\widehat{Q}(x,t)$ describes the slow components. For the fast changing part, let

$$\widetilde{Q}(x,t) = \begin{pmatrix} \widetilde{Q}^1(x,t) & & & \\ & \ddots & & \\ & & \widetilde{Q}^l(x,t) & \\ \widetilde{Q}^{1*}(x,t) & \cdots & \widetilde{Q}^{l*}(x,t) & \widetilde{Q}^*(x,t) \end{pmatrix} \tag{4}$$

such that for each $t \in [0,T]$, and each $k = 1, \ldots, l$, $\widetilde{Q}^k(x,t)$ is a generator with dimension $m_k \times m_k$, $\widetilde{Q}^*(x,t)$ is an $m_* \times m_*$ matrix, $\widetilde{Q}^{k*}(x,t) \in \mathbb{R}^{m_* \times m_k}$. That is, $\widetilde{Q}(x,t)$ represents the strong motion, whereas $\widehat{Q}(x,t)$ together with the diffusions on each torus describes the slow motion and the interaction among each subset $\mathcal{M}_i$, for $i = 1, \ldots, l$, and M_*. The forward differential equation takes the form

$$\begin{aligned}
&\frac{\partial p^\varepsilon}{\partial t} = \frac{1}{\varepsilon} p^\varepsilon \widetilde{Q}(x,t) + p^\varepsilon \widehat{Q}(x,t) + \mathcal{D}^*(x,t) p^\varepsilon, \\
&p^\varepsilon(x,0) = g(x), \qquad g_\alpha(x) \geq 0, \quad \text{for each } \alpha = 1, \ldots, m, \\
&\sum_{\alpha=1}^{m} \int_{[0,1]^r} g_\alpha(x)\, dx = 1.
\end{aligned} \tag{5}$$

Our main interest is on figuring out the asymptotic properties of the solution of (5) when $\varepsilon \to 0$. We obtain the zeroth-order approximation as well as higher-order approximations. To proceed, we need the notion of weak irreducibility of a generator $Q(x,t)$, which was defined in [7].

DEFINITION 2.1. Suppose that $Q(x,t)$ is a generator of the pure jump process. It is said to be weakly irreducible if

$$\begin{aligned}
&\nu(x,t) Q(x,t) = 0, \\
&\nu(x,t) \mathbb{1} = \sum_{i=1}^{m} \nu_i(x,t) = 1
\end{aligned} \tag{6}$$

has a unique solution, and the solution is termed a quasi-stationary distribution.

To carry out the desired asymptotic analysis, we need the following assumptions:

(A1) For each $\alpha = 1, \ldots, m$,

- $a^\alpha(\cdot) \in C^{2,1}$, $b^\alpha(\cdot) \in C^{1,1}$, $g_\alpha(\cdot) \in C^2$;
- $(\partial/\partial t) a^\alpha(x, \cdot)$ and $(\partial/\partial t) b^\alpha(x, \cdot)$ satisfy a Lipschitz condition on $[0, T]$;
- for any $\xi = (\xi_1, \ldots, \xi_r)' \in \mathbb{R}^r$, and $t \in [0, T]$,

$$\sum_{i,j=1}^{r} a_{ij}^\alpha(x,t) \xi_i \xi_j \geq K_0,$$

for some $K_0 > 0$.

(A2) The function $\widetilde{Q}(\cdot,\cdot) \in C^{2,n+2}$. For each

$$x \in [0,1]^r, \ (\partial^{n+2}/\partial t^{n+2}) \widetilde{Q}(x,\cdot)$$

is Lipschitz continuous on $[0, T]$. In addition, $\widehat{Q}(\cdot) \in C^{2,1}$.

(A3) For each $x \in [0,1]^r$, each $t \in [0,T]$, and each $i = 1, \ldots, l$, $\widetilde{Q}^i(x,t)$ is weakly irreducible, and $\widetilde{Q}^*(x,t)$ is Hurwitz, i.e., all of its eigenvalues have negative real parts.

Remark. Since $\widetilde{Q}^*(x,t)$ is Hurwitz, it is stable. In addition, it is nonsingular. An immediate implication of the assumption is that $\mathcal{M}_*$ consists of transient states. Since for fixed x, the jump process cannot have all states being transient (an implication from theory of Markov chains), $\dim \mathcal{M}_* < m$. The strong ellipticity assumed in (A1) ensures the existence of a Green's function of the corresponding parabolic systems (see [2, 9] and the references therein). Note also that in view of (A2), $\widehat{Q}(\cdot)$ is Lipschitz continuous on $[0,1]^r \times [0,T]$.

3. ASYMPTOTIC EXPANSION

We seek asymptotic expansion of $p^\varepsilon(\cdot)$ of the form

$$A_n^\varepsilon(x,t) = \sum_{i=0}^{n} \varepsilon^i u_i(x,t) + \sum_{i=0}^{n} \varepsilon^i w_i(x, t/\varepsilon), \tag{7}$$

where $u_i(\cdot)$ and $w_i(\cdot)$ are termed outer and inner expansions, respectively. To find the outer and inner expansion terms, we follow the route that the actual computation should be done and construct $\{u_i(x,t)\}$ and $\{w_i(x, t/\varepsilon)\}$ recursively. The formal expansion is obtained in this section, whereas the justification and error bounds are provided in the next section.

To obtain the outer expansion, substituting (7) into the forward equation and equating coefficients of ε^i for $i = 0, 1, \ldots, n+2$ lead to

$$\begin{aligned} &u_0(x,t)\widetilde{Q}(x,t) = 0, \\ &u_i(x,t)\widetilde{Q}(x,t) = \frac{\partial u_{i-1}(x,t)}{\partial t} - \mathcal{D}^* u_{i-1}(x,t) - u_{i-1}(x,t)\widehat{Q}(x,t), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } i = 1, \ldots, n. \end{aligned} \tag{8}$$

To get the initial layer expansion, following the usual approach in singular perturbation, define the stretched variable $\tau = t/\varepsilon$. The rationale is that we magnify the time scale for ε near 0 so as to bring out the asymptotics to the foreground. Note that $\widetilde{Q}(\cdot)$ is time-varying. We aim to convert the nonstationary problem to a stationary one. To overcome the difficulty of nonstationarity, we take a Taylor expansion of $\widetilde{Q}(\cdot)$ about $t = 0$. The systems of equations for $\{w_i(\cdot)\}$ are parabolic systems. The corresponding inner expansion terms satisfy

$$
\begin{aligned}
\frac{\partial w_0(x,\tau)}{\partial \tau} &= w_0(x,\tau)\widetilde{Q}(x,0),\\
\frac{\partial w_i(x,\tau)}{\partial \tau} &= w_i(x,\tau)\widetilde{Q}(x,0) + \mathscr{D}^* w_{i-1}(x,\tau) + w_{i-1}(x,\tau)\widehat{Q}(x,\varepsilon\tau)\\
&\quad + \sum_{j=0}^{i-1} \frac{\tau^{j+1}}{(j+1)!} w_{i-j-1}(x,\tau) \frac{\partial^{j+1}\widetilde{Q}(x,0)}{\partial t^{j+1}}.
\end{aligned}
\tag{9}
$$

Introduce the notation of partitioned vectors. For $i = 0, \dots, n+2$, denote

$$
\begin{aligned}
u_i(x,t) &= (u_i^1(x,t), \dots, u_i^l(x,t), u_i^*(x,t)),\\
w_i(x,t) &= (w_i^1(x,t), \dots, w_i^l(x,t), w_i^*(x,t)),
\end{aligned}
\tag{10}
$$

where $u_i^k(x,t), w_i^k(x,t) \in \mathbb{R}^{1\times m_k}$ for $k = 1, \dots, l$, and $u_i^*(x,t), w_i^*(x,t) \in \mathbb{R}^{1\times m_*}$. Our main task in the rest of this section is to find these functions.

3.1. *Finding* $u_0(x,t)$ *and* $w_0(x,\tau)$

The function $u_0(x,t)$ is the limit (as $\varepsilon \to 0$) of the probability density for $t > 0$. Using (8) and (10) yields

$$
\begin{aligned}
u_0^k(x,t)\widetilde{Q}^k(x,t) + u_0^*(x,t)\widetilde{Q}^{*k}(x,t) &= 0,\\
u_0^*(x,t)\widetilde{Q}^*(x,t) &= 0.
\end{aligned}
$$

By virtue of the nonsingularity of $\widetilde{Q}^*(x,t)$, multiplying $[\widetilde{Q}^*(x,t)]^{-1}$ to the last equation above leads to $u_0^*(x,t) = 0'_{m_*} \in \mathbb{R}^{1\times m_*}$. This, in turn, implies that $u_0^k(x,t)\widetilde{Q}^k(x,t) = 0$, for $k = 1, \dots, l$.

Define

$$
\widetilde{\mathbb{1}}(x,t) = \begin{pmatrix}
\mathbb{1}_{m_1} & & & \\
& \ddots & & \\
& & \mathbb{1}_{m_l} & \\
c_1^*(x,t) & \cdots & c_l^*(x,t) & 0_{m_*\times m_*}
\end{pmatrix},
\tag{11}
$$

where

$$
c_k^*(x,t) = -\big[\widetilde{Q}^*(x,t)\big]^{-1}\widetilde{Q}^{k*}(x,t)\,\mathbb{1}_{m_k}, \quad \text{for} \quad k = 1, \dots, l,
$$

and $0_{m_*} \in \mathbb{R}^{m_*\times m_*}$. It is readily verified that $\widetilde{Q}(x,t)\widetilde{\mathbb{1}}(x,t) = 0$, i.e., they are orthogonal.

Assume that $u_0^k(x,t)$ takes the form

$$u_0^k(x,t) = v_0^k(x,t)\nu^k(x,t), \quad \text{for } k = 1, \dots, l, \tag{12}$$

or in a vector form

$$u_0(x,t) = (v_0^1(x,t)\nu^1(x,t), \dots, v_0^l(x,t)\nu^l(x,t), 0'_{m_*}),$$

where $v_0^k(x,t) \in \mathbb{R}$, $v_0(x,t) = (v_0^1(x,t), \dots, v_0^l(x,t), 0) \in \mathbb{R}^{1\times(l+m_*)}$, and $\nu^k(x,t)$ are the quasi-stationary distributions corresponding to $\widetilde{Q}^k(x,t)$. Note that the last subvector $u_0^*(x,t) = 0$ simply reflects the fact that $\mathcal{M}_*$ consists of transient states.

To determine $v_0^k(x,t)$, consider the second equation in (8) with $i = 1$, namely,

$$u_1(x,t)\widetilde{Q}(x,t) = \frac{\partial u_0(x,t)}{\partial t} - \mathscr{D}^* u_0(x,t) - u_0(x,t)\widehat{Q}(x,t). \tag{13}$$

To proceed, we need some notation. Collecting the functions in the definition of operators $\mathscr{D}_\alpha^*$, for $\alpha = 1, \dots, m$, similar to the partition of vectors of $u_i(x,t)$ and $w_i(x,t)$, write

$$(a^1(x,t), \dots, a^m(x,t)) = (a^{\{1\}}(x,t), \dots, a^{\{l\}}(x,t), a^*(x,t)),$$

where for $\alpha = 1, \dots, l$ and $*$,

$$\begin{aligned} a^{\{\alpha\}}(x,t) &= (a^{\{\alpha\},1}(x,t), \dots, a^{\{\alpha\},m_\alpha}(x,t)) \\ &= (a^{\tilde{\alpha}+1}(x,t), \dots, a^{\tilde{\alpha}+m_\alpha}(x,t)), \end{aligned}$$

and where $\tilde{\alpha} = \sum_{\beta=1}^{\alpha-1} m_\beta$. Similarly, define $b^{\{\alpha\}}(x,t)$. In view of (11),

$$\frac{\partial \widetilde{\mathbb{1}}(x,t)}{\partial t} = \begin{pmatrix} 0_{m_1} & & & \\ & \ddots & & \\ & & 0_{m_l} & \\ \dfrac{\partial c_1^*(x,t)}{\partial t} & \cdots & \dfrac{\partial c_l^*(x,t)}{\partial t} & 0_{m_*\times m_*} \end{pmatrix},$$

so

$$v_0(x,t)\mathrm{diag}(\nu^1(x,t), \dots, \nu^l(x,t), 0'_{m_*})\frac{\partial \widetilde{\mathbb{1}}(x,t)}{\partial t} = 0,$$

and

$$\frac{\partial}{\partial t}\big(u_0(x,t)\widetilde{\mathbb{1}}(x,t)\big) = \frac{\partial u_0(x,t)}{\partial t}\widetilde{\mathbb{1}}(x,t).$$

Using (12) and postmultiplying $\widetilde{\mathbb{1}}(x,t)$ in (13) yield that

$$\frac{\partial v_0(x,t)}{\partial t} = \overline{\mathscr{D}}^* v_0(x,t) + v_0(x,t)\overline{Q}(x,t), \tag{14}$$

where

$$\begin{aligned}\overline{Q}(x,t) &= \operatorname{diag}(\nu^1(x,t),\ldots,\nu^l(x,t),0_{m_*\times m_*})\widehat{Q}(x,t)\widetilde{\mathbb{1}}(x,t),\\ \overline{\mathscr{D}}^* v_0(x,t) &= \big(\overline{\mathscr{D}}_1^* v_0^1(x,t),\ldots,\overline{\mathscr{D}}_l^* v^l(x,t),0\big),\end{aligned}$$

and $\overline{\mathscr{D}}^*$ is the adjoint of $\overline{\mathscr{D}}$ with $\overline{\mathscr{D}}_\alpha$ (for $\alpha = 1,\ldots,l$) defined by

$$\begin{aligned}\overline{\mathscr{D}}_\alpha \cdot = {}& \frac{1}{2}\sum_{i,j=1}^{r}\left(\sum_{\beta=1}^{m_\alpha} a_{ij}^{\{\alpha\},\beta}(x,t)\nu_\beta^\alpha(x,t)\right)\frac{\partial^2}{\partial x_i\partial x_j}\cdot\\ &+\sum_{i=1}^{r}\left(\sum_{\beta=1}^{m_\alpha} b_i^{\{\alpha\},\beta}(x,t)\nu_\beta^\alpha(x,t)\right)\frac{\partial}{\partial x_i}\cdot\end{aligned}$$

Equation (14) represents a parabolic system. Choose $v_0^k(x,0) = p^{\varepsilon,k}(x,0)\mathbb{1}_{m_k} = g^k(x)\mathbb{1}_{m_k}$. With such initial data, the solution of (14) is determined, so is the limit (as $\varepsilon \to 0$) of the forward equation for $t > 0$.

Remark. It is worthwhile to note the form of the leading term in the expansion. It has the interpretation of "total probability." That is, for fixed x, it is the probability density of the process belonging to $\mathscr{M}_i$ times probability of actions within $\mathscr{M}_i$ given that the process is in $\mathscr{M}_i$.

Next we proceed with finding the initial layer term $w_0(x,\tau)$. Select the initial condition $w_0(x,0) = g(x) - u_0(x,0)$. It follows that

$$w_0(x,\tau) = w_0(x,0)\exp\big(\widetilde{Q}(x,0)\tau\big). \tag{15}$$

Denote

$$\pi(x,0) = \widetilde{\mathbb{1}}(x,0)\operatorname{diag}(\nu^1(x,0),\ldots,\nu^l(x,0),0_{m_*\times m_*}).$$

The following lemma holds, whose proof is in the Appendix.

LEMMA 3.1. *Under conditions* (A1)–(A3), *there are positive constants K and κ such that for $\tau \ge 0$,*

$$\sup_{x\in[0,1]^r}\left|\exp\big(\widetilde{Q}(x,0)\tau\big) - \pi(x,0)\right| \le K\exp(-\kappa\tau).$$

Notice that $w_0(x,0)$ is orthogonal to $\pi(x,0)$. By virtue of Lemma 3.1,

$$\begin{aligned}\sup_{x\in[0,1]^r}|w_0(x,\tau)| &= \left|w_0(x,0)\exp\big(\widetilde{Q}(x,0)\tau\big)\right|\\ &= \sup_{x\in[0,1]^r}\Big|w_0(x,0)\big[\exp\big(\widetilde{Q}(x,0)\tau\big)-\pi(x,0)\big]\\ &\qquad\qquad + w_0(x,0)\pi(x,0)\Big|\\ &\le K\exp(-\kappa\tau).\end{aligned}$$

That is, $w_0(x,\tau)$ decays exponentially fast.

3.2. *Determining $u_i(x,t)$ and $w_i(x,t/\varepsilon)$*

The subject matter of this subsection is to determine $u_i(x,t)$ and $w_i(x,\tau)$ for $i=1,\dots,n+2$. Although we aim to obtain an nth-order expansion, two more terms are needed to deduce the approximation error bound. To see the pattern, first consider $u_1(x,t)$. Since $u_0(x,t)$ has been obtained, the right-hand side of (8) for $i=1$ is a completely known function. Denote it by $\psi_0(x,t)$. Then the nonhomogeneous equation is

$$u_1(x,t)\widetilde{Q}(x,t)=\psi_0(x,t).$$

Assume that $u_1(x,t)$ has the form

$$u_1(x,t)=(v_1^1(x,t)\nu^1(x,t),\dots,v_1^l(x,t)\nu^l(x,t),0'_{m_*})+U_1(x,t),$$

where $U_1(x,t)$ is a particular solution of the nonhomogeneous system and $v_1^k(x,t)\in\mathbb{R}$, for $k=1,\dots,l$.

Setting $i=2$ in (8),

$$u_2(x,t)\widetilde{Q}(x,t)=\frac{\partial u_1(x,t)}{\partial t}-\mathscr{D}^*u_1(x,t)-u_1(x,t)\widehat{Q}(x,t).$$

It is a nonhomogeneous system. Owing to the well-known Fredholm alternative, it has a solution when the right-hand side is orthogonal to $\widetilde{\mathbb{1}}(x,t)$ which is a solution of the adjoint homogeneous equation $\widetilde{Q}(x,t)z(x,t)=0$. Postmultiplying by $\widetilde{\mathbb{1}}(x,t)$ leads to

$$\begin{aligned}\frac{\partial v_1(x,t)}{\partial t} &= \overline{\mathscr{D}}^*v_1(x,t)+v_1(x,t)\overline{Q}(x,t)+(\mathscr{D}^*U_1(x,t))\widetilde{\mathbb{1}}(x,t)\\ &\quad + U_1(x,t)\widehat{Q}(x,t)\widetilde{\mathbb{1}}(x,t)-\left(\frac{\partial U_1(x,t)}{\partial t}\right)\widetilde{\mathbb{1}}(x,t),\end{aligned}\tag{16}$$

with known $U_1(x,t)$, the solution of (16) is determined when the initial condition is specified, which will be obtained from the matching condition of $u_1(x,t)$ with that of initial layer term $w_1(x,\tau)$, namely,

$$u_1(x,0)+w_1(x,0)=0\quad\text{or}\quad w_1(x,0)=-u_1(x,0).\tag{17}$$

Turning to (9), the equation

$$\frac{\partial w_1(x,\tau)}{\partial \tau} = w_1(x,\tau)\widetilde{Q}(x,0) + w_0(x,\tau)\widehat{Q}(x,\varepsilon\tau) + \mathscr{D}^* w_0(x,\tau) + \tau w_0(x,\tau)\frac{\partial \widetilde{Q}(x,0)}{\partial t}$$

yields that

$$\begin{aligned} w_1(x,\tau) &= w_1(x,0)\exp\big(\widetilde{Q}(x,0)\tau\big) \\ &\quad + \int_0^\tau w_0(x,0)\exp\big(\widetilde{Q}(x,0)s\big)\widehat{Q}(x,\varepsilon s) \\ &\qquad \times \exp\big(\widetilde{Q}(x,0)(\tau-s)\big)\,ds \\ &\quad + \int_0^\tau [\mathscr{D}^* w_0(x,s)]\exp\big(\widetilde{Q}(x,0)(\tau-s)\big)\,ds \\ &\quad + \int_0^\tau s w_0(x,0)\exp\big(\widetilde{Q}(x,0)s\big)\frac{\partial \widetilde{Q}(x,0)}{\partial t} \\ &\qquad \times \exp\big(\widetilde{Q}(x,0)(\tau-s)\big)\,ds. \end{aligned} \tag{18}$$

Once the initial condition of $w_1(x,0)$ is specified, the function $w_1(x,\tau)$ and hence $u_1(x,t)$ are determined completely.

By virtue of Lemma 3.1, $\lim_{\tau\to\infty}\exp(\widetilde{Q}(x,0)\tau) = \pi(x,0)$. As for the second term of the right-hand side of (18), owing to (A2),

$$\sup_{(x,t)\in[0,1]^r\times[0,T]} \big|\widehat{Q}(x,t)\big| \le K_T < \infty,$$

where K_T is a positive constant. Consequently, by the orthogonality of $w_0(x,0)$ to $\pi(x,0)$,

$$\begin{aligned} &\left|\int_0^\tau w_0(x,0)\exp\big(\widetilde{Q}(x,0)s\big)\widehat{Q}(x,\varepsilon s)\,ds\right| \\ &\quad = \left|\int_0^\tau w_0(x,0)\big(\exp\big(\widetilde{Q}(x,0)s\big) - \pi(x,0)\big)\widehat{Q}(x,\varepsilon s)\,ds\right| \\ &\quad \le \int_0^\tau \big|w_0(x,0)\big(\exp\big(\widetilde{Q}(x,0)s\big) - \pi(x,0)\big)\big| \sup_{(x,t)\in[0,1]^r\times[0,T]}\big|\widehat{Q}(x,t)\big|\,ds \\ &\quad \le K\int_0^\infty \exp(-\kappa s)\,ds < \infty. \end{aligned}$$

Since

$$\widehat{Q}(x,\varepsilon s) = \widehat{Q}(x,0) + \big[\widehat{Q}(x,\varepsilon s) - \widehat{Q}(x,0)\big],$$

and in view of the Lipschitz continuity of $\widehat{Q}(\cdot)$ and the orthogonality of $w_0(x,0)$ to $\pi(x,0)$,

$$\begin{aligned}
&\left|\int_0^\tau w_0(x,0)\exp\big(\widetilde{Q}(x,0)s\big)\big(\widehat{Q}(x,\varepsilon s)-\widehat{Q}(x,0)\big)\,ds\right| \\
&\qquad \le K\varepsilon\int_0^\tau s\big|w_0(x,0)\big[\big(\exp\big(\widetilde{Q}(x,0)s\big)-\pi(x,0)\big)\big]\big|\,ds \\
&\qquad \le K\varepsilon\int_0^\infty s\exp(-\kappa s)\,ds = O(\varepsilon).
\end{aligned}$$

Therefore, as $\varepsilon \to 0$ or equivalently $\tau \to \infty$, we have

$$\begin{aligned}
&\lim_{\tau\to\infty}\int_0^\tau w_0(x,0)\exp\big(\widetilde{Q}(x,0)s\big)\widehat{Q}(x,\varepsilon s)\exp\big(\widetilde{Q}(x,0)(\tau-s)\big)ds \\
&\qquad = \int_0^\infty w_0(x,0)\exp\big(\widetilde{Q}(x,0)s\big)\widehat{Q}(x,0)\,ds\,\pi(x,0),
\end{aligned}$$

and the improper integral converges. Similarly,

$$\begin{aligned}
&\lim_{\tau\to\infty}\int_0^\tau sw_0(x,0)\exp\big(\widetilde{Q}(x,0)s\big)\frac{\partial\widetilde{Q}(x,0)}{\partial t}\exp\big(\widetilde{Q}(x,0)(\tau-s)\big)\,ds \\
&\qquad = \int_0^\infty sw_0(x,0)\exp\big(\widetilde{Q}(x,0)s\big)\frac{\partial\widetilde{Q}(x,0)}{\partial t}\,ds\,\pi(x,0),
\end{aligned}$$

and the latter integration makes sense.

To proceed, we need an a priori estimate for the term $\mathscr{D}^* w_0(x,\tau)$. We state a lemma below. A sketch of its proof is in the Appendix.

LEMMA 3.2. *Under* (A1)–(A3), *the following assertions hold*:

(1) *Suppose that* $\varphi(\cdot)$ *is a solution of*

$$\begin{aligned}
&\frac{\partial\varphi(x,\tau)}{\partial\tau} = \varphi(x,\tau)\widetilde{Q}(x,0) \quad \textit{with} \\
&\varphi(x,0) = H(x) \quad \textit{such that} \\
&H(x)\widetilde{\mathbb{1}}(x,0) = 0,
\end{aligned} \tag{19}$$

where $H(\cdot)\in C^1$. *Then for some* $\kappa>0$ *and* $K>0$,

$$\sup_{x\in[0,1]^r}\big|(\partial/\partial x)\varphi(x,\tau)\big| \le K\exp(-\kappa\tau).$$

(2) *Suppose* $\varphi(x,\tau) = (\varphi^1(x,\tau),\ldots,\varphi^l(x,\tau),\varphi^*(x,\tau))$ *is a solution of the nonhomogeneous system*

$$\begin{aligned}
&\frac{\partial\varphi(x,\tau)}{\partial\tau} = \varphi(x,\tau)\widetilde{Q}(x,0)+\xi(x,\tau) \quad \textit{with} \\
&\varphi(x,0) = H(x).
\end{aligned} \tag{20}$$

Denote $\psi(x,\tau)=(\partial/\partial x)\psi(x,\tau)$. *Assume* $\xi(\cdot)\in C^{1,1}$, $\xi(x,\tau)$ *decays exponentially fast together with* $(\partial/\partial x)\xi(x,\tau)$, $\xi^k(x,\tau)=\xi^{k,1}(x,\tau)+\xi^{k,2}(x,\tau)$, *and* $H(\cdot)\in C^1$ *such that* $(\partial/\partial x)\xi^{k,1}(x,\tau)\mathbb{1}_{m_k}=0$ *and*

$$\begin{aligned}
&H^k(x)\mathbb{1}_{m_k}\nu^k(x,0)+\int_0^\infty \psi^*(x,s)\widetilde{Q}^{k*}(x,0)\,ds\,\mathbb{1}_{m_k}\nu^k(x,0)\\
&\quad+\int_0^\infty \varphi^*(x,s)\frac{\partial\widetilde{Q}^{k*}(x,0)}{\partial x}\,ds\,\mathbb{1}_{m_k}\nu^k(x,0)\\
&\quad+\int_0^\infty \frac{\partial\xi^{k,2}(x,\tau)}{\partial x}\,ds\,\mathbb{1}_{m_k}\nu^k(x,0)=0.
\end{aligned}\tag{21}$$

Then

$$\sup_{x\in[0,1]^r}\left|\frac{\partial\varphi(x,\tau)}{\partial x}\right|\le K\exp(-\kappa\tau).$$

With repeated applications of Lemma 3.2, $\mathscr{D}^*w_0(x,\tau)$ decays exponentially fast,

$$\lim_{\tau\to\infty}\int_0^\tau \mathscr{D}^*w_0(x,s)\exp\big(\widetilde{Q}(x,0)(\tau-s)\big)\,ds=\int_0^\infty \mathscr{D}^*w_0(x,s)\,ds\,\pi(x,0),$$

and the integral on the right-hand side makes sense. As a result,

$$w_1(x,0)\pi(x,0)=-\overline{w}_0(x)\pi(x,0),\tag{22}$$

where

$$\begin{aligned}
\overline{w}_0(x,0)=&\int_0^\infty w_0(x,0)\exp\big(\widetilde{Q}(x,0)s\big)\,ds\,\widehat{Q}(x,0)\\
&+\int_0^\infty sw_0(x,0)\exp\big(\widetilde{Q}(x,0)s\big)\frac{\partial\widetilde{Q}(x,0)}{\partial t}\,ds\\
&+\int_0^\infty \mathscr{D}^*w_0(x,s)\,ds.
\end{aligned}\tag{23}$$

Denote

$$\overline{w}_0(x,0)=(\overline{w}_0^1(x,0),\dots,\overline{w}_0^l(x,0),\overline{w}_0^*(x,0)),$$

where $\overline{w}_0^k(x,0)\in\mathbb{R}^{1\times m_k}$ and $\overline{w}_0^*(x,0)\in\mathbb{R}^{1\times m_*}$. Using the definition of $\pi(x,0)$, it follows from (22),

$$w_1^k(x,0)\mathbb{1}_{m_k}+w_1^*(x,0)c_k^*(x,0)=-\big(\overline{w}_0^k(x,0)\mathbb{1}_{m_k}+\overline{w}_0^*(x,0)c_k^*(x,0)\big),$$

for $k=1,\dots,l$. Noticing that $u_1^*(x,0)+w_1^*(x,0)=0$, the above can be further written as

$$w_1^k(x,0)\mathbb{1}_{m_k}=u_1^*(x,0)c_k^*(x,0)-\big(\overline{w}_0^k(x,0)\mathbb{1}_{m_k}+\overline{w}_0^*(x,0)c_k^*(x,0)\big).$$

The $u_1^*(x,0)$ is not known yet; we proceed to find it. In view of (8) with $i=1$,

$$u_1^k(x,t)\widetilde{Q}^k(x,t) + u_1^*(x,t)\widetilde{Q}^{*1}(x,t) = \psi_0^k(x,t), \quad \text{for } k = 1,\dots,l,$$
$$u_1^*(x,t)\widetilde{Q}^*(x,t) = \psi_0^*(x,t),$$

where $\psi_0(x,t) = (\psi_0^1(x,t),\dots,\psi_0^l(x,t),\psi_0^*(x,t))$ is the partitioned vector for $\psi_0(x,t)$. It is readily seen that $u_1^*(x,t) = \psi_0^*(x,t)[\widetilde{Q}^*(x,t)]^{-1}$ due to the nonsingularity of $\widetilde{Q}^*(x,t)$. Thus

$$\begin{aligned} w_1^k(x,0)\mathbb{1}_{m_k} &= \psi_0^*(x,0)[\widetilde{Q}^*(x,0)]^{-1}c_k^*(x,0) \\ &\quad - \big(\overline{w}_0^k(x,0)\mathbb{1}_{m_k} + \overline{w}_0^*(x,0)c_k^*(x,0)\big), \end{aligned} \tag{24}$$

and hence $v_1^k(x,0)$ can be determined uniquely; so can $u_1(x,t)$ and $w_1(x,t/\varepsilon)$.

In view of the limit results obtained thus far,

$$\begin{aligned} w_1(x,\tau) = {}& w_1(x,0)\big(\exp\big(\widetilde{Q}(x,0)\tau\big) - \pi(x,0)\big) \\ &+ \int_0^\tau w_0(x,0)\exp\big(\widetilde{Q}(x,0)s\big)\widehat{Q}(x,\varepsilon s) \\ &\qquad \times \big(\exp\big(\widetilde{Q}(x,0)(\tau-s)\big) - \pi(x,0)\big)\,ds \\ &+ \int_0^\tau \mathscr{D}^* w_0(x,s)\big(\exp\big(\widetilde{Q}(x,0)(\tau-s)\big) - \pi(x,0)\big)\,ds \\ &+ \int_0^\tau s w_0(x,0)\exp\big(\widetilde{Q}(x,0)s\big)\frac{\partial\widetilde{Q}(x,0)}{\partial t} \\ &\qquad \times \big(\exp\big(\widetilde{Q}(x,0)(\tau-s)\big) - \pi(x,0)\big)\,ds \\ &- \int_\tau^\infty w_0(x,0)\exp\big(\widetilde{Q}(x,0)s\big)\widehat{Q}(x,\varepsilon s)\pi(x,0)\,ds \\ &- \int_\tau^\infty \mathscr{D}^* w_0(x,s)\pi(x,0)\,ds \\ &- \int_\tau^\infty s w_0(x,0)\exp\big(\widetilde{Q}(x,0)s\big)\frac{\partial\widetilde{Q}(x,0)}{\partial t}\pi(x,0)\,ds \\ &+ \big(w_1(x,0)\pi(x,0) + \overline{w}_0(x,0)\pi(x,0)\big). \end{aligned}$$

The last line above is 0. The rest of the terms on the right side of the equality sign all decay exponentially fast by virtue of Lemma 3.1 and Lemma 3.2. It is then easily verified that

$$\sup_{x\in[0,1]^r} |w_1(x,\tau)| \le K\exp(-\kappa_0\tau).$$

Using exactly the same idea, we proceed to find higher-order outer and inner expansions. For $i \geq 1$, write the second equation in (8) as

$$\begin{aligned} u_i(x,t)\widetilde{Q}(x,t) &= \psi_{i-1}(x,t) \\ &\stackrel{\text{def}}{=} \frac{\partial u_{i-1}(x,t)}{\partial t} - \mathscr{D}^* u_{i-1}(x,t) - u_{i-1}(x,t)\widehat{Q}(x,t). \end{aligned}$$

By virtue of the notation of partitioned vector, we further write the above as

$$\begin{aligned} u_i^k(x,t)\widetilde{Q}^k(x,t) + u_i^*(x,t)\widetilde{Q}^{*k}(x,t) &= \psi_{i-1}^k(x,t), \\ u_i^*(x,t)\widetilde{Q}^*(x,t) &= \psi_{i-1}^*(x,t). \end{aligned} \tag{25}$$

It is readily seen that $u_i^*(x,t) = \psi_{i-1}^*(x,t)[\widetilde{Q}^*(x,t)]^{-1}$, and hence it can be treated as a known subvector. For each $i = 1, \ldots, n+2$, assume $u_i(x,t)$ to be of the form

$$u_i(x,t) = (v_i^1(x,t)\nu^1(x,t), \ldots, v_i^l(x,t)\nu^l(x,t), 0'_{m_*}) + U_i(x,t),$$

where $v_i^k(x,t) \in \mathbb{R}$ for $k = 1, \ldots, l$, and $U_i(x,t)$ is a particular solution of the nonhomogeneous system (25). For u_{i+1} in the defining equation (8), multiplying through by $\widetilde{\mathbb{1}}(x,t)$ gives us

$$\begin{aligned} \frac{\partial v_i(x,t)}{\partial t} &= \overline{\mathscr{D}}^* v_i(x,t) + v_i(x,t)\overline{Q}(x,t) + (\mathscr{D}^* U_i(x,t))\widetilde{\mathbb{1}}(x,t) \\ &\quad + U_i(x,t)\widehat{Q}(x,t)\widetilde{\mathbb{1}}(x,t) - \frac{\partial U_i(x,t)}{\partial t}\widetilde{\mathbb{1}}(x,t). \end{aligned} \tag{26}$$

To determine the initial condition $v_i(x,0)$, we work with the initial layer term $w_i(x,\tau)$. Using (9) and keeping in mind that $w_j(x,\tau)$ have been found for $j = 1, \ldots, i-1$,

$$\begin{aligned} w_i(x,\tau) &= w_i(x,0)\exp\big(\widetilde{Q}(x,0)\tau\big) \\ &\quad + \int_0^\tau \mathscr{D}^* w_{i-1}(x,s)\exp\big(\widetilde{Q}(x,0)(\tau-s)\big)\,ds \\ &\quad + \int_0^\tau w_{i-1}(x,s)\widehat{Q}(x,\varepsilon s)\exp\big(\widetilde{Q}(x,0)(\tau-s)\big)\,ds \\ &\quad + \int_0^\tau \sum_{j=0}^{i-1} \frac{s^{j+1}}{(j+1)!} w_{i-j-1}(x,s)\frac{\partial^{j+1}\widetilde{Q}(x,0)}{\partial t^{j+1}} \\ &\qquad \times \exp\big(\widetilde{Q}(x,0)(\tau-s)\big)\,ds, \end{aligned} \tag{27}$$

with $w_i(x,0) = -u_i(x,0)$.

Analogous to the derivation for $w_0(x, 0)$, $w_j(x, \tau)$ decays exponentially fast, and by virtue of Lemma 3.2, $\mathscr{D}^* w_{i-1}(x, \tau)$ decays exponentially fast. Define

$$\begin{aligned}\overline{w}_{i-1}(x, 0) = \int_0^\infty & w_{i-1}(x, s)\widehat{Q}(x, 0)\, ds \\ &+ \int_0^\infty \mathscr{D}^* w_{i-1}(x, s)\, ds \\ &+ \int_0^\infty \sum_{j=0}^{i-1} \frac{s^{j+1}}{(j+1)!} w_{i-j-1}(x, s) \frac{\partial^{j+1}\widetilde{Q}(x, 0)}{\partial t^{j+1}}\, ds,\end{aligned}$$

and denote

$$\overline{w}_{i-1}(x, 0) = (\overline{w}_{i-1}^1(x, 0), \ldots, \overline{w}_{i-1}^l(x, 0), \overline{w}_{i-1}^*(x, 0)).$$

We have for each $k = 1, \ldots, l$,

$$\begin{aligned}w_i^k(x, 0)\mathbb{1}_{m_k} = \psi_{i-1}^*(x, 0)\big[\widetilde{Q}^*(x, 0)\big]^{-1} c_k^*(x, 0) \\ - (\overline{w}_{i-1}^k(x, 0)\mathbb{1}_{m_k} + \overline{w}_{i-1}^*(x, 0)c_k^*(x, 0)).\end{aligned} \tag{28}$$

Hence $v_i^k(x, 0)$ is determined and so are $u_i(x, t)$ and $w_i(x, t/\varepsilon)$. We summarize the construction in the following proposition.

PROPOSITION 3.3. *Under the conditions of* (A1)–(A3), *the asymptotic expansion* (7) *can be constructed such that* $\{u_i(x, t)\}$ *is sufficiently smooth and*

$$\sup_{x\in[0,1]^r} |w_i(x, t/\varepsilon)| \le K \exp(-\kappa_0 t/\varepsilon),$$

for $i = 1, \ldots, n+2$.

4. ASYMPTOTIC ERROR BOUNDS

In this section, we prove the validity of the asymptotic expansion. Denote the forward and backward operators by

$$\begin{aligned}L^{\varepsilon,*}\cdot &= -\frac{\partial\cdot}{\partial t} + \cdot Q^\varepsilon + \mathscr{D}^*\cdot, \\ L^\varepsilon\cdot &= \frac{\partial\cdot}{\partial s} + Q^\varepsilon\cdot + \mathscr{D}\cdot,\end{aligned} \tag{29}$$

where $Q^\varepsilon(x, t)$ is as defined in (3). Recall the definition of the approximation sequence (7). Subsection 4.1 focuses on the error bounds of

$L^{\varepsilon,*}A_k^\varepsilon(x,t)$, Subsection 4.2 derives an auxiliary result via stochastic representation of solution of the corresponding partial differential equations, and Subsection 4.3 concludes the proof of the desired asymptotic bounds.

4.1. *Bounds on* $L^{\varepsilon,*}A_k^\varepsilon$

For $i = 1, \ldots, n+2$, define a sequence of approximation errors $e_i^\varepsilon(x,t)$ for each $(x,t) \in [0,1]^r \times [0,T]$ by

$$e_i^\varepsilon(x,t) = p^\varepsilon(x,t) - A_i^\varepsilon(x,t). \tag{30}$$

We have the following lemma.

LEMMA 4.1. *Under Conditions* (A1)–(A3),

$$\sup_{(x,t)\in[0,1]^r\times[0,T]} |L^{\varepsilon,*}e_i^\varepsilon(x,t)| = O(\varepsilon^i) \quad \textit{for } i \le n+2.$$

Proof. We estimate $L^{\varepsilon,*}e_i^\varepsilon(x,t)$, where $L^{\varepsilon,*}$ is defined by (29). Note that $L^{\varepsilon,*}p^\varepsilon = 0$, so $L^{\varepsilon,*}e_i^\varepsilon(x,t) = -L^{\varepsilon,*}A_i^\varepsilon(x,t)$.

For $i = 1$,

$$\begin{aligned}
-L^{\varepsilon,*}A_1^\varepsilon(x,t) &= -\big(L^{\varepsilon,*}u_0(x,t) + \varepsilon L^{\varepsilon,*}u_1(x,t) \\
&\qquad + L^{\varepsilon,*}w_0(x,t/\varepsilon) + \varepsilon L^{\varepsilon,*}w_1(x,t/\varepsilon)\big) \\
&= \frac{\partial u_0(x,t)}{\partial t} - \frac{1}{\varepsilon}u_0(x,t)\widetilde{Q}(x,t) \\
&\quad - u_0(x,t)\widehat{Q}(x,t) - \mathscr{D}^* u_0(x,t) \\
&\quad + \varepsilon\frac{\partial u_1(x,t)}{\partial t} - u_1(x,t)\widetilde{Q}(x,t) \\
&\quad - \varepsilon u_1(x,t)\widehat{Q}(x,t) - \varepsilon\mathscr{D}^* u_1(x,t) \\
&\quad + \frac{\partial w_0(x,t/\varepsilon)}{\partial t} - \frac{1}{\varepsilon}w_0(x,t/\varepsilon)\widetilde{Q}(x,t) \\
&\quad - w_0(x,t/\varepsilon)\widehat{Q}(x,t) - \mathscr{D}^* w_0(x,t/\varepsilon) \\
&\quad + \varepsilon\frac{\partial w_1(x,t/\varepsilon)}{\partial t} - w_1(x,t/\varepsilon)\widetilde{Q}(x,t) \\
&\quad - \varepsilon w_1(x,t/\varepsilon)\widehat{Q}(x,t) - \varepsilon\mathscr{D}^* w_1(x,t/\varepsilon).
\end{aligned} \tag{31}$$

In view of the defining equation (8),

$$\begin{aligned}\sup_{(x,t)\in[0,1]^r\times[0,T]}\bigg|&\frac{\partial u_0(x,t)}{\partial t}-\frac{1}{\varepsilon}u_0(x,t)\widetilde{Q}(x,t)-u_0(x,t)\widehat{Q}(x,t)\\&-\mathscr{D}^*u_0(x,t)+\varepsilon\frac{\partial u_1(x,t)}{\partial t}-u_1(x,t)\widetilde{Q}(x,t)\\&-\varepsilon u_1(x,t)\widehat{Q}(x,t)-\varepsilon\mathscr{D}^*u_1(x,t)\bigg|\\=\sup_{(x,t)\in[0,1]^r\times[0,T]}&\left|\varepsilon\frac{\partial u_1(x,t)}{\partial t}-\varepsilon u_1(x,t)\widehat{Q}(x,t)-\varepsilon\mathscr{D}^*u_1(x,t)\right|\\=O(\varepsilon).&\end{aligned}$$

Owing to the exponential decay of $w_1(\cdot)$ and Lemma 3.2, $\mathscr{D}^*w_1(\cdot)$ is bounded and

$$\sup_{(x,t)\in[0,1]^r\times[0,T]}|\varepsilon\mathscr{D}^*(x,t)w_1(x,t/\varepsilon)|=O(\varepsilon).$$

Note that for each $i=1,\ldots,n+2$,

$$t^i\exp(-\kappa_0t/\varepsilon)=(t/\varepsilon)^i\exp(-\kappa_0t/\varepsilon)\varepsilon^i=O(\varepsilon^i).$$

Therefore,

$$\begin{aligned}&\left|w_0(x,t/\varepsilon)\left[\widetilde{Q}(x,0)+t\widetilde{Q}^{(1)}(x,0)-\widetilde{Q}(x,t)\right]\right|\\&\qquad\le Kt^2\exp(-\kappa_0t/\varepsilon)=O(\varepsilon^2),\end{aligned}$$

and

$$\left|w_1(x,t/\varepsilon)\left[\widetilde{Q}(x,0)-\widetilde{Q}(x,t)\right]\right|\le Kt\exp(-\kappa_0t/\varepsilon)=O(\varepsilon),$$

and the bounds hold uniformly in $(x,t)\in[0,1]^r\times[0,T]$. As a result,

$$\begin{aligned}&\frac{\partial w_0(x,t/\varepsilon)}{\partial t}-\frac{1}{\varepsilon}w_0(x,t/\varepsilon)\widetilde{Q}(x,t)-w_0(x,t/\varepsilon)\widehat{Q}(x,t)\\&\quad-\mathscr{D}^*w_0(x,t/\varepsilon)+\varepsilon\frac{\partial w_1(x,t/\varepsilon)}{\partial t}-w_1(x,t/\varepsilon)\widetilde{Q}(x,t)\\&\quad-\varepsilon w_1(x,t/\varepsilon)\widehat{Q}(x,t)-\varepsilon\mathscr{D}^*w_1(x,t/\varepsilon)\\&=\frac{1}{\varepsilon}w_0(x,t/\varepsilon)\left[\widetilde{Q}(x,0)+t\widetilde{Q}^{(1)}(x,0)-\widetilde{Q}(x,t)\right]\\&\quad+w_1(x,t/\varepsilon)\left[\widetilde{Q}(x,0)-\widetilde{Q}(x,t)\right]\\&\quad-\varepsilon w_1(x,t/\varepsilon)\widehat{Q}(x,t)-\varepsilon\mathscr{D}^*w_1(x,t/\varepsilon)\\&=O(\varepsilon).\end{aligned}$$

Combining the estimates above,

$$\sup_{(x,t)\in[0,1]^r\times[0,T]} |L^{\varepsilon,*} e_1^\varepsilon(x,t)| = O(\varepsilon).$$

The same kind of estimate yields the estimates for $i = 1, \ldots, n+2$. We provide the argument for $i = n+2$ below, i.e., we show that

$$\sup_{(x,t)\in[0,1]^r\times[0,T]} |L^{\varepsilon,*} A_{n+2}^\varepsilon(x,t)| = O(\varepsilon^{n+2}).$$

Note that

$$\begin{aligned}
-L^{\varepsilon,*} & A_{n+2}^\varepsilon(x,t) \\
&= -\biggl(\sum_{i=0}^{n+2} \varepsilon^i L^{\varepsilon,*} u_i(x,t) + \sum_{i=0}^{n+2} \varepsilon^i L^{\varepsilon,*} w_i(x,t/\varepsilon)\biggr) \\
&= \sum_{i=0}^{n+2} \varepsilon^i \biggl(\frac{\partial u_i(x,t)}{\partial t} - \frac{1}{\varepsilon} u_i(x,t)\widetilde{Q}(x,t) \\
&\qquad\qquad - u_i(x,t)\widehat{Q}(x,t) - \mathscr{D}^*(x,t)u_i(x,t)\biggr) \\
&\quad + \sum_{i=0}^{n+2} \varepsilon^i \biggl(\frac{\partial w_i(x,t/\varepsilon)}{\partial t} - \frac{1}{\varepsilon} w_i(x,t/\varepsilon)\widetilde{Q}(x,t) \\
&\qquad\qquad - w_i(x,t/\varepsilon)\widehat{Q}(x,t) - \mathscr{D}^*(x,t)w_i(x,t/\varepsilon)\biggr).
\end{aligned} \tag{32}$$

In view of (8),

$$\begin{aligned}
\sum_{i=0}^{n+1} \varepsilon^i \biggl(& \frac{\partial u_i(x,t)}{\partial t} - u_{i+1}(x,t)\widetilde{Q}(x,t) \\
& - u_i(x,t)\widehat{Q}(x,t) - \mathscr{D}^*(x,t)u_i(x,t)\biggr) = 0
\end{aligned}$$

and

$$u_0(x,t)\widetilde{Q}(x,t) = 0.$$

Consequently,

$$\begin{aligned}
\sup_{(x,t)\in[0,1]^r\times[0,T]} & \biggl|\sum_{i=0}^{n+2} \varepsilon^i L^{\varepsilon,*} u_i(x,t)\biggr| \\
&= \sup_{(x,t)\in[0,1]^r\times[0,T]} \biggl|\varepsilon^{n+2} \frac{\partial u_{n+2}(x,t)}{\partial t} - \varepsilon^{n+2} u_{n+2}(x,t)\widehat{Q}(x,t) \\
&\qquad\qquad - \varepsilon^{n+2}\mathscr{D}^*(x,t)u_{n+2}(x,t)\biggr| \\
&= O(\varepsilon^{n+2}).
\end{aligned} \tag{33}$$

For the initial layer terms, using the same kind of estimates as that of $e_1^{\varepsilon}(\cdot)$, detailed estimate reveals that

$$\sup_{(x,t)\in[0,1]^r\times[0,T]} \left| \sum_{i=0}^{n+2} \varepsilon^i \left(\frac{\partial w_i(x,t/\varepsilon)}{\partial t} - \frac{1}{\varepsilon} w_i(x,t/\varepsilon)\widetilde{Q}(x,t) \right.\right.$$

$$\left.\left. - w_i(x,t/\varepsilon)\widehat{Q}(x,t) - \mathscr{D}^*(x,t) w_i(x,t/\varepsilon) \right) \right|$$

$$= O(\varepsilon^{n+2}). \tag{34}$$

Combining (33) and (34), $\sup_{(x,t)\in[0,1]^r\times[0,T]} |L^{\varepsilon,*} A_{n+2}^{\varepsilon}(x,t)| = O(\varepsilon^{n+2})$. The lemma thus follows. ∎

4.2. *Bounds via Stochastic Representation*

Here we follow the approach of [5]. To determine the desired bounds, we use a stochastic representation of the corresponding partial differential equations. The proof of Lemma 4.2 is essentially a combination of [2, Chapters II and III]) and Ito's formula ([3] or [4, Chapter 2.5]). The proof of Lemma 4.3 is very similar to Proposition 4.1 of [5], and is thus omitted.

LEMMA 4.2. *Assume* (A1)–(A3). *Suppose that the column-vector-valued function*

$$\eta(x,s) = (\eta(x,1,s), \ldots, \eta(x,m,s)) \in \mathbb{R}^{m\times 1}$$

is a solution of

$$\begin{aligned} L^{\varepsilon}\eta &= \frac{\partial \eta}{\partial s} + \mathscr{D}\eta + Q^{\varepsilon}\eta = \varphi(x,s), \quad \textit{for } s < t, \\ \eta(x,t) &= 0 \quad \textit{for } x \in [0,1]^r, \end{aligned} \tag{35}$$

where $\varphi(x,s) = (\varphi(x,1,s), \ldots, \varphi(x,m,s))$ *satisfies* $|\varphi(x,s)| \le K(t-s)^{-1/2}$. *Denote* $Y^{\varepsilon,i}(t) = (X^{\varepsilon}(t), \gamma^{\varepsilon}(t))$ *to be the process* $Y^{\varepsilon}(t)$ *starts at* $X^{\varepsilon}(s) = x$ *with* $\gamma^{\varepsilon}(s) = i$ *for* $i = 1, \ldots, m$. *Then the solution admits a stochastic representation*

$$\eta_i(x,s) = -E\int_s^t \varphi(Y^{\varepsilon,i}(v), v)\, dv. \tag{36}$$

LEMMA 4.3. *Suppose that the conditions of Lemma* 4.2 *are satisfied and that* $\Delta^{\varepsilon}(\cdot)$ *is a function continuous on* $[0,1]^r \times [0,T]$, *satisfying* $\sup_{(x,t)\in[0,1]^r\times[0,T]} |\Delta^{\varepsilon}(x,t)| = O(\varepsilon^{i+1})$ *for some* i. *Suppose* $f^{\varepsilon}(\cdot)$ *is the solution to*

$$\begin{aligned} L^{\varepsilon,*} f^{\varepsilon} &= \Delta^{\varepsilon}(x,t), \\ f^{\varepsilon}(x,0) &= 0 \quad \textit{for all } x \in [0,1]^r. \end{aligned} \tag{37}$$

Then

$$\sup_{(x,t)\in[0,1]^r\times[0,T]} |f^{\varepsilon}(x,t)| = O(\varepsilon^i). \tag{38}$$

4.3. *Bounds on* $p^\varepsilon(x,t) - A_n^\varepsilon(x,t)$

We begin with the following lemma. The proof uses a "back tracking" argument.

PROPOSITION 4.4. *Under Conditions* (A1)–(A3),

$$\sup_{(x,t)\in[0,1]^r\times[0,T]} |e_i^\varepsilon(x,t)| = O(\varepsilon^{i+1}) \quad \textit{for } i = 0, 1, \ldots, n.$$

Proof. Due to Lemma 4.1, $L^{\varepsilon,*} e_2^\varepsilon(x,t) = O(\varepsilon^2)$ uniformly in (x,t). By virtue of Lemma 4.3,

$$\sup_{(x,t)\in[0,1]^r\times[0,T]} |e_2^\varepsilon(x,t)| = O(\varepsilon). \tag{39}$$

Since

$$\begin{aligned} e_2^\varepsilon(x,t) = e_0^\varepsilon(x,t) &- \varepsilon u_1(x,t) - \varepsilon w_1(x,t/\varepsilon) \\ &- \varepsilon^2 u_2(x,t) - \varepsilon^2 w_2(x,t/\varepsilon), \end{aligned} \tag{40}$$

the smoothness of $u_j(\cdot)$ and the exponential decay of $w_j(\cdot)$ for $j = 1, 2$ then imply

$$\sup_{(x,t)\in[0,1]^r\times[0,T]} |\varepsilon u_1(x,t) + \varepsilon w_1(x,t/\varepsilon)| = O(\varepsilon),$$

$$\sup_{(x,t)\in[0,1]^r\times[0,T]} |\varepsilon^2 u_2(x,t) + \varepsilon^2 w_2(x,t/\varepsilon)| = O(\varepsilon^2).$$

Equations (39) and (40) together with the estimates above infer

$$\sup_{(x,t)\in[0,1]^r\times[0,T]} |e_0^\varepsilon(x,t)| = O(\varepsilon).$$

Similarly, for each $i = 1, \ldots, n$, Lemma 4.1 infers that

$$\sup_{(x,t)\in[0,1]^r\times[0,T]} |L^{\varepsilon,*} e_{i+2}^\varepsilon(x,t)| = O(\varepsilon^{i+2})$$

and hence by Lemma 4.3,

$$\sup_{(x,t)\in[0,1]^r\times[0,T]} |e_{i+2}^\varepsilon(x,t)| = O(\varepsilon^{i+1}).$$

However,

$$\begin{aligned} e_{i+2}^\varepsilon(x,t) = e_i^\varepsilon(x,t) &- \varepsilon^{i+1} u_{i+1}(x,t) - \varepsilon^{i+1} w_{i+1}(x,t/\varepsilon) \\ &- \varepsilon^{i+2} u_{i+2}(x,t) - \varepsilon^{i+2} w_{i+2}(x,t/\varepsilon). \end{aligned}$$

In view of the smoothness of $u_j(x,t)$ and the exponential decay of $w_j(x,t/\varepsilon)$ for $j = i+1, i+2$, $\sup_{(x,t)\in[0,1]^r\times[0,T]} |e_i^\varepsilon(x,t)| = O(\varepsilon^{i+1})$. The desired estimate thus follows. ∎

We summarize the results of Propositions 3.3 and 4.4 in the theorem below.

THEOREM 4.5. *Under conditions* (A1)–(A3), *the sequence* $A_n^{\varepsilon}(x,t)$ (*equivalently, two sequences* $\{u_i(x,t)\}$ *and* $\{w_i(x,t/\varepsilon)\}$) *defined in* (7) *can be constructed such that for each* $i \le n$, $u_i(x,t)$ *is sufficiently smooth,* $w_i(x,t/\varepsilon)$ *decays exponentially fast, and*

$$\sup_{(x,t)\in[0,1]^r\times[0,T]} |p^{\varepsilon}(x,t) - A_n^{\varepsilon}(x,t)| = O(\varepsilon^{n+1}).$$

5. FURTHER REMARKS

We have developed asymptotic expansion of a class of switching diffusion processes. The main effort has been on treating fast switching processes, where the fast changing generator takes the form (4). Similar to the study in pure jump processes in [15], one may study alternative forms of generators such as generators corresponding states that are all recurrent and/or generators including absorbing states. Fast diffusion may also be treated. One of models of interest has the generator of the form

$$L^{*,\varepsilon} f = -\frac{\partial f}{\partial t} + f\widetilde{Q} + \varepsilon f\widehat{Q} + \frac{1}{\varepsilon}\mathscr{D}^* f.$$

This paper focuses on deterministic aspects of the processes involved. Our current effort is devoted to the stochastic aspects such as what are the limit processes for various aggregation of the singularly perturbed Markov processes.

APPENDIX

Proof of Lemma 3.1. The proof is similar to that of [15, Lemma 6.22] with slight modifications. We include it here for completeness.

Let

$$z(x,\tau) = (z^1(x,\tau), \ldots, z^l(x,\tau), \qquad z^*(x,\tau)) \in \mathbb{R}^{1\times m},$$

be a solution to

$$\frac{\partial z(x,\tau)}{\partial \tau} = z(x,\tau)\widetilde{Q}(x,0), \qquad z(x,0) = \widehat{z}(x),$$

where $\widehat{z}(x) = (\widehat{z}^1(x), \ldots, \widehat{z}^l(x), \widehat{z}^*(x))$ is an arbitrary initial condition. Then

$$z^*(x,\tau) = \widehat{z}^*(x)\exp\big(\widetilde{Q}^*(x,0)\tau\big)$$

and

$$z^k(x,\tau)=\widehat{z}^k(x)\exp\big(\widetilde{Q}^k(x,0)\tau\big) + \int_0^\tau z^*(x,s)\widetilde{Q}^{k*}(x,0)\exp\big(\widetilde{Q}^k(x,0)(\tau-s)\big)\,ds,$$

for $k=1,\dots,l$.

It is enough to show that for all $\widehat{z}(x)$,

$$\sup_{x\in[0,1]^r}\Big|\widehat{z}(x)\big(\exp\big(\widetilde{Q}(x,0)\tau\big)-\pi(x,0)\big)\Big| \le K \sup_{x\in[0,1]^r}\big|\widehat{z}(x)\big|e^{-\kappa_0\tau}.$$

In fact, for each $k=1,\dots,l$, we have

$$\begin{aligned}
&z^k(x,\tau)-\bigg(\widehat{z}^k(x)\,\mathbb{1}_{m_k}\nu^k(x,0)\\
&\qquad + \widehat{z}^*(x)\int_0^\infty \exp\big(\widetilde{Q}^*(x,0)s\big)\,ds\,\widetilde{Q}^{k*}(x,0)\,\mathbb{1}_{m_k}\nu^k(x,0)\bigg)\\
&\quad=\widehat{z}^k(x)\big(\exp\big(\widetilde{Q}^k(x,0)\tau\big)-\mathbb{1}_{m_k}\nu^k(x,0)\big)\\
&\qquad+\widehat{z}^*(x)\int_0^\tau \exp\big(\widetilde{Q}^*(x,0)s\big)\widetilde{Q}^{k*}(x,0)\\
&\qquad\times\big(\exp\big(\widetilde{Q}^k(x,0)(\tau-s)\big)-\mathbb{1}_{m_k}\nu^k(x,0)\big)\,ds\\
&\qquad-\widehat{z}^*(x)\int_\tau^\infty \exp\big(\widetilde{Q}^*(x,0)s\big)\widetilde{Q}^{k*}(x,0)\,\mathbb{1}_{m_k}\nu^k(x,0)\,ds.
\end{aligned}\tag{41}$$

Since $\widetilde{Q}^*(x,0)$ is Hurwitz, the last term above is bounded above by $K|\widehat{z}^*(x)|\exp(-\widehat{\kappa}_{l+1}\tau)$. It can be shown (see, e.g., [15, Appendix] and the references therein) that for some $\widehat{\kappa}_k>0$,

$$\sup_{x\in[0,1]^r}\big|\exp\big(\widetilde{Q}^k(x,0)\tau\big)-\mathbb{1}_{m_k}\nu^k(x,0)\big|\le K\exp\big(-\widehat{\kappa}_k\tau\big).$$

Choose $\kappa_0=\min_{1\le k\le l+1}\widehat{\kappa}_k$. The terms in the second and the third lines of (41) are bounded by $K\exp(-\kappa_0\tau)$. The desired estimate thus follows. ∎

Proof of Lemma 3.2. The proof uses the same idea as that of Lemma 3.1. We outline the main steps below.

Step 1: Differentiating (19) w.r.t. x and denoting $\psi(x,\tau)=(\partial/\partial x)\varphi(x,\tau)$ lead to

$$\frac{\partial\psi^k(x,\tau)}{\partial\tau}=\psi^k(x,\tau)\widetilde{Q}^k(x,0)+\psi^*(x,\tau)\widetilde{Q}^{k*}(x,0) + \varphi^k(x,\tau)\frac{\partial\widetilde{Q}^k(x,0)}{\partial x}+\varphi^*(x,\tau)\frac{\partial\widetilde{Q}^{k*}(x,0)}{\partial x}\tag{42}$$

for $k = 1, \ldots, l$ and

$$\frac{\partial \psi^*(x,\tau)}{\partial \tau} = \psi^*(x,\tau)\widetilde{Q}^*(x,0) + \varphi^*(x,\tau)\frac{\partial \widetilde{Q}^*(x,0)}{\partial x}. \tag{43}$$

Therefore,

$$\begin{aligned}\psi^*(x,\tau) &= \psi^*(x,0)\exp\big(\widetilde{Q}^*(x,0)\tau\big)\\ &\quad + \int_0^\tau \varphi^*(x,s)\frac{\partial \widetilde{Q}^*(x,0)}{\partial x}\exp\big(\widetilde{Q}^*(x,0)(\tau-s)\big)\,ds, \end{aligned}\tag{44}$$

and

$$\begin{aligned}\psi^k(x,\tau) &= \psi^k(x,0)\exp\big(\widetilde{Q}^k(x,0)\tau\big)\\ &\quad + \int_0^\tau \psi^*(x,s)\widetilde{Q}^{k*}(x,0)\exp\big(\widetilde{Q}^k(x,0)(\tau-s)\big)\,ds\\ &\quad + \int_0^\tau \varphi^k(x,s)\frac{\partial \widetilde{Q}^k(x,0)}{\partial x}\exp\big(\widetilde{Q}^k(x,0)(\tau-s)\big)\,ds\\ &\quad + \int_0^\tau \varphi^*(x,s)\frac{\partial \widetilde{Q}^{k*}(x,0)}{\partial x}\exp\big(\widetilde{Q}^k(x,0)(\tau-s)\big)\,ds. \end{aligned}\tag{45}$$

Since $\widetilde{Q}^*(x,0)$ is Hurwitz, it follows from (44) $\sup_{x\in[0,1]^r}|\psi^*(x,\tau)| \le K\exp(-\kappa\tau)$. Therefore, we need only show $\psi^k(x,\tau)$ decays exponentially fast for $k = 1, \ldots, l$.

For each $k = 1, \ldots, l$, define

$$\begin{aligned}\widetilde{\Psi}^k(x) &= \psi^k(x,0)\mathbb{1}_{m_k}\nu^k(x,0) + \int_0^\infty \psi^*(x,s)\,ds\,\widetilde{Q}^{k*}(x,0)\mathbb{1}_{m_k}\nu^k(x,0)\\ &\quad + \int_0^\infty \varphi^k(x,s)\frac{\partial \widetilde{Q}^k(x,0)}{\partial x}\,ds\,\mathbb{1}_{m_k}\nu^k(x,0)\\ &\quad + \int_0^\infty \varphi^*(x,s)\frac{\partial \widetilde{Q}^{k*}(x,0)}{\partial x}\mathbb{1}_{m_k}\nu^k(x,0). \end{aligned}\tag{46}$$

Similar to the proof of Lemma 3.1, for each $k = 1, \ldots, l$,

$$\sup_{x\in[0,1]^r}\big|\psi^k(x,\tau) - \widetilde{\Psi}^k(x)\big| \le K\exp(-\kappa\tau).$$

Owing to the orthogonality $H(x)\widetilde{\mathbb{1}}(x,0) = \varphi(x,0)\widetilde{\mathbb{1}}(x,0) = 0$ and the definition of $\widetilde{\mathbb{1}}(x,0)$,

$$\begin{aligned}0 &= \frac{\partial H^k(x)}{\partial x}\mathbb{1}_{m_k} + \frac{\partial H^*(x)}{\partial x}c_k^*(x,0) + H^*(x)\big[\widetilde{Q}^*(x,0)\big]^{-1}\\ &\quad \times\left(\frac{\partial \widetilde{Q}^*(x,0)}{\partial x}\big[\widetilde{Q}^*(x,0)\big]^{-1}\widetilde{Q}^{k*}(x,0) - \frac{\partial \widetilde{Q}^{k*}(x,0)}{\partial x}\right)\mathbb{1}_{m_k}. \end{aligned}\tag{47}$$

By virtue of (44),

$$\begin{aligned}
&\int_0^\infty \psi^*(x,s)\widetilde{Q}^{k*}(x,0)\,ds\,\mathbb{1}_{m_k}\nu^k(x,0)\\
&\quad=\int_0^\infty \psi^*(x,0)\exp\bigl(\widetilde{Q}^*(x,0)s\bigr)\,ds\,\widetilde{Q}^{k*}(x,0)\,\mathbb{1}_{m_k}\nu^k(x,0)\\
&\qquad+\int_0^\infty\int_0^s \varphi^*(x,0)\exp\bigl(\widetilde{Q}^*(x,0)t\bigr)\frac{\partial\widetilde{Q}^*(x,0)}{\partial x}\\
&\qquad\qquad\times\exp\bigl(\widetilde{Q}^*(x,0)(s-t)\bigr)\,dt\,\widetilde{Q}^{k*}(x,0)\,ds\,\mathbb{1}_{m_k}\nu^k(x,0).
\end{aligned}$$

It is easily seen that

$$\begin{aligned}
&\int_0^\infty \psi^*(x,0)\exp\bigl(\widetilde{Q}^*(x,0)s\bigr)\,ds\,\widetilde{Q}^{k*}(x,0)\,\mathbb{1}_{m_k}\nu^k(x,0)\\
&\qquad=-\psi^*(x,0)\bigl[\widetilde{Q}^*(x,0)\bigr]^{-1}\widetilde{Q}^{k*}(x,0)\,\mathbb{1}_{m_k}\nu^k(x,0)\\
&\qquad=\frac{\partial H^k(x)}{\partial x}c_k^*(x,0)\nu^k(x,0),
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\infty\int_0^s \varphi^*(x,0)\exp\bigl(\widetilde{Q}^*(x,0)t\bigr)\frac{\partial\widetilde{Q}^*(x,0)}{\partial x}\exp\bigl(\widetilde{Q}^*(x,0)(s-t)\bigr)\,dt\\
&\quad=\lim_{\tau\to\infty}H^*(x)\int_0^\tau \exp\bigl(\widetilde{Q}^*(x,0)t\bigr)\frac{\partial\widetilde{Q}^*(x,0)}{\partial x}\\
&\qquad\times\int_t^\tau \exp\bigl(\widetilde{Q}^*(x,0)(s-t)\bigr)\,ds\,\widetilde{Q}^{k*}(x,0)\,\mathbb{1}\,\nu^k(x,0)\,dt\\
&\quad=H^*(x)\bigl[\widetilde{Q}^*(x,0)\bigr]^{-1}\frac{\partial\widetilde{Q}^*(x,0)}{\partial x}\bigl[\widetilde{Q}^*(x,0)\bigr]^{-1}\widetilde{Q}^{k*}(x,0)\,\mathbb{1}_{m_k}\nu^k(x,0).
\end{aligned}$$

Similarly, it can be shown that

$$\begin{aligned}
&\int_0^\infty \varphi^*(x,s)\frac{\partial\widetilde{Q}^{k*}(x,0)}{\partial x}\,ds\,\mathbb{1}_{m_k}\nu^k(x,0)\\
&\qquad=-H^*(x)\bigl[\widetilde{Q}^*(x,0)\bigr]^{-1}\frac{\partial\widetilde{Q}^{k*}(x,0)}{\partial x}\,\mathbb{1}_{m_k}\nu^k(x,0).
\end{aligned}$$

These estimates, (47), and the orthogonality condition $(\partial/\partial x)\widetilde{Q}^k(x,0)\mathbb{1}_{m_k}=0$ then imply

$$\sup_{x\in[0,1]^r}|\psi(x,\tau)|\le K\exp(-\kappa\tau).$$

To prove the second assertion of Lemma 3.2, write the solution in terms of the partitioned vectors for each k. Again, it is readily seen that $\psi^*(x,\tau)$

decays exponentially fast. As for $\psi^k(x, \tau)$ for $k = 1, \ldots, l$, in lieu of (47), utilize $\xi^k(x, \tau) = \xi^{k,1}(x, \tau) + \xi^{k,2}(x, \tau)$ and (21), use the same techniques with the addition of the nonhomogeneous term, and proceed as in the previous case. This completes the proof of the lemma. ∎

REFERENCES

1. M. H. A. Davis, "Markov Models and Optimization," Chapman & Hall, London, 1993.
2. S. D. Eidelman, "Parabolic Systems," North-Holland, New York, 1969.
3. I. I. Gihman and A. V. Skorohod, "Theory of Stochastic Processes, III," Springer-Verlag, Berlin, 1979.
4. N. Ikeda and S. Watanabe, "Stochastic Differential Equations and Diffusion Processes," 2nd ed., Kodansha/North-Holland, Tokyo, 1989.
5. A. M. Il'in, R. Z. Khasminskii, and G. Yin, Singularly perturbed switching diffusions: rapid switchings and fast diffusions, to appear in J. Optim. Theory. Appl., 1998.
6. R. Z. Khasminskii and G. Yin, Asymptotic series for singularly perturbed Kolmogorov–Fokker–Planck equations, *SIAM J. Appl. Math.* **56** (1996), 1766–1793.
7. R. Z. Khasminskii, G. Yin, and Q. Zhang, Asymptotic expansions of singularly perturbed systems involving rapidly fluctuating Markov chains, *SIAM J. Appl. Math.* **56** (1996), 277–293.
8. R. Z. Khasminskii, G. Yin, and Q. Zhang, Constructing asymptotic series for probability distribution of Markov chains with weak and strong interactions, *Quart. Appl. Math.* **LV** (1997), 177–200.
9. O. A. Ladyzhenskaia, V. A. Solonnikov, and N. Ural'tseva, Linear and quasi-linear equations of parabolic type, Trans. Math. Monographs, Vol. 23, Amer. Math. Soc., Providence, RI, 1968.
10. Z. G. Pan and T. Başar, H^∞-control of Markovian jump linear systems and solutions to associated piecewise-deterministic differential games, *in* "New Trends in Dynamic Games and Applications," (G. J. Olsder, Ed.), Birkhäuser, Boston, 1995, pp. 61–94.
11. R. G. Phillips and P. V. Kokotovic, A singular perturbation approach to modelling and control of Markov chains, *IEEE Trans. Automat. Control* **26** (1981), 1087–1094.
12. S. P. Sethi and Q. Zhang, "Hierarchical Decision Making in Stochastic Manufacturing Systems," Birkhäuser, Boston, 1994.
13. A. V. Skorohod, Asymptotic methods of the theory of stochastic differential equations, Trans. Math. Monographs, Vol. 78, Amer. Math. Soc., Providence, RI, 1989.
14. D. W. Stroock and S. R. S. Varadhan, "Multidimensional Diffusion Processes," Springer-Verlag, Berlin, 1979.
15. G. Yin and Q. Zhang, "Continuous-time Markov Chains and Applications: A Singular Perturbation Approach," Applications of Mathematics, Vol. 37, Springer-Verlag, New York, 1998.
16. G. Yin and Q. Zhang, Eds., Mathematics of stochastic manufacturing systems, Proc. 1996 AMS-SIAM Summer Seminar in Applied Mathematics, Lectures in Applied Mathematics, Amer. Math. Soc., Providence, RI, 1997.
17. Q. Zhang and G. Yin, A central limit theorem for singularly perturbed nonstationary finite state Markov chains, *Ann. Appl. Probab.* **6**, 650–670.
18. Q. Zhang and G. Yin, Structural properties of Markov chains with weak and strong interactions, *Stochastic Process Appl.* **70** (1997), 181–197.